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DESIGN EFFICIENCY UNDER MODEL UNCERTAINTY FOR NONREGULAR FRACTIONS OF GENERAL FACTORIALS

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Abstract: A criterion of design efficiency, under model uncertainty, is studied with reference to possibly nonregular fractions of general factorials. The criterion is expressed in terms of the departure of the design from being an orthogonal array of strength three or four. A Kronecker calculus for factorial arrangements facilitates the derivation. The results are followed by a numerical study and the findings are compared with those based on other design criteria.

Key words and phrases: Generalized minimum aberration, Kronecker calculus, minimum moment aberration, orthogonal array, projection.

1. Introduction

Recently, Cheng, Deng and Tang (2002), hereafter abbreviated CDT, reported results on design efficiency, under model uncertainty, for nonregular fractions of two-level factorials. Their criterion concerns models that include the general mean, all main effects and a selection of two-factor interactions (2fi's) and, in the absence of prior knowledge on which 2fi's are active, it considers the average performance of a design over all possible models with the same number of 2fi's. As discussed by these authors, this approach is in the spirit of the criterion of estimation capacity introduced by Sun (1993), and studied by Cheng, Steinberg and Sun (1999), Cheng and Mukerjee (1998, 2001) and Mukerjee, Chan and Fang (2000) for regular fractions.

The present article aims at extending the work of CDT on design efficiency to general factorials including the asymmetrical ones. This calls for a substantial modification of their mathematical techniques since, unlike in the two-level case, each factorial effect may no longer be represented by a single treatment contrast. A Kronecker calculus for factorial arrangements facilitates the formulation of the model matrices as well as the derivation of the key results. The main results are presented in Section 2 where we also indicate the connection with the departure of the design from being an orthogonal array of various strengths. This, in turn, entails a link with the generalized minimum aberration (GMA) criterion (Tang and Deng (1999) and Xu and Wu (2001)). In Section 3, the present criterion is applied to 18-run nonregular fractions of 2×3^3 and 2×3^4 factorials. The findings are seen to be in agreement with those according to the GMA criterion and the minimum moment aberration (MMA) criterion (Xu (2003)). Proofs appear in the appendix.

2. Main Results

Suppose there are m factors $F_1,\ldots,F_m$ at $s_1,\ldots,s_m(\geq 2)$ levels respectively. For $1\leq j\leq m$, the levels of F_j are coded as $0,\ldots,s_j-1$. Consider a possibly nonregular fraction or design consisting of the treatment combinations $a_{i1}a_{i2}\cdots a_{im}$, $1\leq i\leq N$, where $a_{ij}\in\{0,\ldots,s_j-1\}$ for every i,j. Throughout, N is fixed and it is supposed that these N treatment combinations, when written as rows, form an orthogonal array (OA) of strength two.

We assume the absence of interactions involving three or more factors. Note that altogether there are $W(=m(m-1)/2)$ 2fi's. For $1\leq w\leq W$, let $H(w)$ be the collection of all sets of w 2fi's. For any $h\in H(w)$, let $M(h)$ be the model consisting of only the general mean, all main effects and the w 2fi's in h, and $X(h)$ be the model matrix under $M(h)$. The matrix $X(h)$ consists of blocks of columns that correspond to the general mean and the factorial effects in $M(h)$. The blocks of columns associated with the 2fi's are related to those associated with the main effects via Kronecker products. A detailed expression for $X(h)$ appears in (A.2) in the Appendix. As usual, it is assumed that the observational errors are homoscedastic and uncorrelated.

Under $M(h)$, the D-criterion aims at maximizing $\det\{X(h)^TX(h)\}$. If one wishes to include w 2fi's in the model, but has no prior knowledge on which w should be included, then it makes sense to consider the average of $\det\{X(h)^TX(h)\}$ over all $h\in H(w)$. This is the D_w-criterion of CDT. However, it is difficult to handle this criterion algebraically. On the other hand, minimization of $\mathrm{tr}[\{X(h)^TX(h)\}^2]$ is a good surrogate for the maximization of $\det\{X(h)^TX(h)\}$. This happens because $\mathrm{tr}\,\{X(h)^TX(h)\}$ is the same for all designs under consideration; cf., (A.2), (A.4) and (A.13) in the appendix. Consequently, a large $\det\{X(h)^TX(h)\}$ is typically accompanied by a small $\mathrm{tr}[\{X(h)^TX(h)\}^2]$ since both occur when the eigenvalues of $X(h)^TX(h)$ are close to one another (the same argument shows that minimization of $\mathrm{tr}[\{X(h)^TX(h)\}^2]$ would be a good surrogate also if one worked with the A- or E-criteria). Hence following CDT, we consider the design criterion

$$E_w=\binom{W}{w}^{-1}\sum_{h\in H(w)}tr[\{X(h)^TX(h)\}^2], \tag{2.1}$$

and aim at studying designs that keep E_w small for every w, especially for smaller values of w which are more relevant under effect sparsity.

Lemma 2.1, presented below and proved in the appendix, gives an expression for E_w which is useful both algebraically and numerically. Some more notation will help. For any distinct j,k,l $(1 \le j,k,l \le m)$, let $n^{jkl}_{\alpha\beta\gamma}$ be the number of times the factors F_j, F_k and F_l appear at levels α, β and γ respectively among the N treatment combinations in the design, and define

$$\phi(jkl) = s_j s_k s_l \sum\sum\sum \left(n^{jkl}_{\alpha\beta\gamma}\right)^2, \tag{2.2}$$

where the triple sum is over $0 \le \alpha \le s_j - 1$, $0 \le \beta \le s_k - 1$, $0 \le \gamma \le s_l - 1$. Similarly, for any distinct j,k,l,u $(1 \le j,k,l,u \le m)$, define the quantities $n^{jklu}_{\alpha\beta\gamma\rho}$, and hence $\phi(jklu)$, exactly along the lines of (2.2). Let $\Delta(3)$ be the set of all ordered triplets jkl, where $1 \le j < k < l \le m$, and $\Delta(4)$ be the set of all ordered four-tuples $jklu$, where $1 \le j < k < l < u \le m$. Finally, in (2.4) below and the rest of this paper, a "constant" may depend on $w, N, m, s_1, \ldots, s_m$ but is the same for all designs.

Lemma 2.1. *For $1 \le w \le W$, with*

$$E^*_w = \sum_{jkl\in\Delta(3)} \left(6 + \frac{2(w-1)}{W-1}(s_j + s_k + s_l - 3m + 3)\right)\phi(jkl) + \frac{6(w-1)}{W-1}\sum_{jklu\in\Delta(4)} \phi(jklu), \tag{2.3}$$

$$E_w = \text{constant} + (w/W)E^*_w,\ 1 \le w \le W. \tag{2.4}$$

In view of (2.4), hereafter we consider the quantities E^*_w. By (2.3), for $3 \le w \le W$,

$$E^*_w = E^*_1 + (w-1)(E^*_2 - E^*_1), \tag{2.5}$$

a fact which is useful for computational purposes. Lemma 2.1 also helps in expressing E^*_w, and hence E_w, in terms of measures of the departure of the design from being represented by an OA of strength three or four. To that effect, some more notation is introduced.

For $1 \le j \le m$, let

$$V_j(0) = s_j^{-1} 1_j 1_j^T, \qquad V_j(1) = I_j - s_j^{-1} 1_j 1_j^T, \tag{2.6}$$

where 1_j is the $s_j \times 1$ vector with all elements unity and I_j is the identity matrix of order s_j. For any binary m-tuple $x = x_1 \cdots x_m$, define the matrix

$$V(x) = V_1(x_1) \otimes V_2(x_2) \otimes \cdots \otimes V_m(x_m), \tag{2.7}$$

where $\otimes$ denotes the Kronecker product. Let $\nu = \prod_{j=1}^{m} s_j$, and n be the $\nu \times 1$ vector whose elements represent the replication numbers of the ν treatment combinations in the design, arranged in the lexicographic order. For $1 \leq g \leq m$, let

$$B_g = \sum_{x \in \Omega(g)} n^T V(x) n, \tag{2.8}$$

where $\Omega(g)$ is the set of binary m-tuples with exactly g 1's.

Clearly, $B_g \geq 0$ for every g, as the matrices $V(x)$ are nonnegative definite. Since the treatment combinations in the design form an OA of strength two, by (2.6)$-$(2.8), $B_1 = B_2 = 0$. Similarly, it can be seen that the design is represented by an OA of strength three if and only if $B_3 = 0$, and an OA of strength four if and only if, in addition, $B_4 = 0$. Hence, as argued by Fang, Ma and Mukerjee (2002) (see also Tang (2001)), B_3 is a natural measure of the departure of the design from being represented by an OA of strength three, whereas B_4 measures the additional departure of the design from an OA of strength four. It can also be seen that

$$B_g = \nu^{-1} N^2 A_g, \ 1 \leq g \leq m, \tag{2.9}$$

where $(A_1, \ldots, A_m)$ is the generalized wordlength pattern (GWP) of the design (Tang and Deng (1999) and Xu and Wu (2001)). Theorem 2.1 below expresses E_w^* in terms of B_3, B_4 and the related quantities $B(jkl)$, where for $jkl \in \Delta(3)$, $B(jkl) = n^T V(x(jkl))n$, with $x(jkl)$ being the binary m-tuple that has 1 in the jth, kth and lth positions and 0 elsewhere. Again, $B(jkl)$ is nonnegative and equals zero if and only if the projection of the design onto the three factors F_j, F_k and F_l is an OA of strength three. The proof of Theorem 2.1 appears in the appendix.

Theorem 2.1. *For* $1 \leq w \leq W$,

$$\begin{aligned} E_w^* &= \textit{constant} \\ &\quad + 6\nu\Big[B_3 + \frac{w-1}{W-1}\Big(B_4 - 2B_3 + \frac{1}{3}\sum_{jkl \in \Delta(3)} (s_j + s_k + s_l) B(jkl)\Big)\Big]. \end{aligned} \tag{2.10}$$

Remark 2.1. By (2.8),

$$B_3 = \sum_{jkl \in \Delta(3)} B(jkl). \tag{2.11}$$

Hence if $s_1 = \cdots = s_m (= s$, say) then (2.10) simplifies to

$$E_w^* = \text{constant} + 6\nu \left[\left(1 + \frac{w-1}{W-1}(s-2)\right) B_3 + \frac{w-1}{W-1} B_4 \right]. \tag{2.12}$$

The coefficient of B_3 in (2.12) is much larger than that of B_4, especially for relatively smaller values of w. Hence a design that sequentially minimizes $B_3, B_4, \ldots$ (recall that $B_1 = B_2 = 0$ for any design considered) should perform well under the criterion considered here. Therefore, strengthening the findings of CDT, from (2.9) it follows that for general symmetrical factorials a GMA design should have an edge over others under the present criterion as well.

Remark 2.2. For two-level factorials, by (2.9) and (2.12), $E_w^* =$ constant $+\, 6N^2[A_3 + \{(w-1)/(W-1)\}A_4]$ which, in conjunction with (2.4), is in agreement with CDT.

Remark 2.3. For asymmetrical factorials, by (2.9) and (2.10), $E_1^* =$ constant $+\, 6N^2A_3$. While the link between $E_w^*, w \geq 2$, and the GWP is less obvious, the numerical study in Section 3 suggests that the GMA criterion tends to be in agreement with the present one.

Remark 2.4. Interestingly, $B(jkl)$ actually occurs in (2.10) only for $w \geq 2$ and not for $w = 1$. If two or more 2fi's are included in the model, then any two of them can potentially involve a common factor. Such common factors contribute to the term involving $B(jkl)$ in (2.10). The same happens with the coefficient of $\phi(jkl)$ in (2.3). Equation (A.15) in the appendix and the discussion preceding it make this explicit.

3. A Numerical Study

Table 7C.2 of Wu and Hamada (2000) shows an OA$(18,2^1 3^7)$ of strength two, with 18 rows and 8 columns, where the first column has two symbols and the remaining columns have three symbols each. Consideration of the first column together with any three other columns of this array yields a nonregular fraction of a 2×3^3 factorial in 18 treatment combinations. Any such design, given by the first, jth, kth and lth columns, is denoted by $1jkl$ $(2 \leq j < k < l \leq 8)$. For any of the 35 possible designs so obtained, a simple counting of degrees of freedom reveals that $X(h)^T X(h)$ is singular whenever the model involves five or more 2fi's. Hence, we consider E_w^* only for $1 \leq w \leq 4$. It is seen that the collection of these 35 designs can be partitioned into six classes, as shown in Table 3.1, such that all designs in the same class have the same E_w^* for every w and also the same GWP. Table 3.2 shows $E_w^*, 1 \leq w \leq 4$, and the GWP against these six classes. Equations (2.3) and (2.4) facilitate these computations. From Table 3.2, it is clear that, for every w, the ranking of designs according to E_w^* is precisely the same as that according to the GMA criterion. In fact, it can be seen that this ranking is also the same as that under the MMA criterion, with natural weights, as based on the first five moments.

The phenomenon of identical ranking of designs according to the E_w^* and the criteria of GMA and MMA continues to hold if one instead considers designs for a 2×3^4 factorial that arise in a similar manner from the OA($18,2^1 3^7$) mentioned above. The details are omitted here. This suggests that even for asymmetrical factorials the latter two criteria are good surrogates for the present criterion which has a direct statistical meaning.

Table 3.1. Equivalent classes of designs for a 2×3^3 factorial arising from an OA($18,2^1 3^7$).

Class	Designs
1	1248, 1258, 1367, 1458
2	1236, 1237, 1267
3	1234, 1235, 1246, 1247, 1256, 1257
4	1238, 1268, 1278
5	1345, 1346, 1347, 1348, 1356, 1357, 1358, 1368, 1378, 1456, 1457, 1467, 1468, 1478, 1567, 1568, 1578, 1678
6	1245

Table 3.2. Values of E_w^* and GWP for the six classes of designs.

Class	E_1^*	E_2^*	E_3^*	E_4^*	GWP
1	8748.0	9525.6	10303.2	11080.8	(0, 0, 1/2, 3/2)
2	9720.0	10497.6	11275.2	12052.8	(0, 0, 1, 1)
3	10044.0	10735.2	11426.4	12117.6	(0, 0, 7/6, 5/6)
4	11016.0	11707.2	12398.4	13089.6	(0, 0, 5/3, 1/3)
5	11340.0	11944.8	12549.6	13154.4	(0, 0, 11/6, 1/6)
6	11664.0	12441.6	13219.2	13996.8	(0, 0, 2, 0)

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Appendix

Proof of Lemma 2.1. For $1 \leq j \leq m$, let 1_j be the $s_j \times 1$ vector with all elements unity, I_j^* be the identity matrix of order $s_j - 1$, and $P_j = [p_j(0), \ldots, p_j(s_j - 1)]$

be a matrix, of order $(s_j-1)\times s_j$, satisfying

$$P_jP_j^T = s_jI_j^*, \qquad P_j1_j = 0. \tag{A.1}$$

Also, let Z_j be the $N\times(s_j-1)$ matrix with rows $p_j(a_{ij})^T, 1\le i\le N$. Furthermore, for $1\le j<k\le m$, let Z_{jk} be the $N\times\{(s_j-1)(s_k-1)\}$ matrix with rows $\{p_j(a_{ij})\otimes p_k(a_{ik})\}^T$, $1\le i\le N$. Then following Mukerjee (1999) (see also Xu and Wu (2001) and Cheng and Ye (2004)),

$$X(h) = [1_{(N)}, Z_1,\ldots,Z_m,\ldots,Z_{jk},\ldots], \tag{A.2}$$

where $1_{(N)}$ is the $N\times 1$ vector with all elements unity, and any Z_{jk} is included in $X(h)$ if and only if the 2fi F_jF_k belongs to h. In (A.2), $1_{(N)}$ corresponds to the general mean, any Z_j corresponds to the main effect of F_j and any Z_{jk} corresponds to the 2fi F_jF_k. With all factors at two levels, one can take $P_j = [-1\ \ 1]$ in view of (A.1), and then (A.2) agrees with CDT. For general factorials, the specific choice of the matrices P_j, subject to (A.1), does not affect our results.

Since the treatment combinations in the design form an OA of strength two, the following hold as a consequence of (A.1) and the definitions of Z_j and Z_{jk}:

$$Z_j^T1_{(N)} = 0\ (1\le j\le m),\ Z_{jk}^T1_{(N)} = 0\ (1\le j<k\le m), \tag{A.3}$$
$$Z_j^TZ_j = NI_j^*\ (1\le j\le m),\ Z_j^TZ_k = 0\ (1\le j\ne k\le m). \tag{A.4}$$

Hence by (A.2),

$$\begin{aligned}\mathrm{tr}[\{X(h)^TX(h)\}^2] = N^2\Big[1+\sum_{j=1}^m(s_j-1)\Big] &+ 2\sum_{j=1}^m\sum_{kl\in h}\mathrm{tr}(Z_j^TZ_{kl}Z_{kl}^TZ_j)\\ &+\sum_{jk\in h}\sum_{lu\in h}\mathrm{tr}(Z_{jk}^TZ_{lu}Z_{lu}^TZ_{jk}),\end{aligned} \tag{A.5}$$

where $\sum_{kl\in h}$ denotes sum over all kl $(k<l)$ such that $F_kF_l\in h$, $\sum_{jk\in h}$ and $\sum_{lu\in h}$ are similarly defined.

Let $\Delta(2)$ be the set of all ordered pairs kl, where $1\le k<l\le m$, and for any $kl\in\Delta(2)$, define $H(w;kl)$ as the collection of all sets of w 2fi's that contain F_kF_l. Clearly, $H(w;kl)$ has cardinality $\binom{W-1}{w-1}$. Hence

$$\begin{aligned}\sum_{h\in H(w)}\sum_{j=1}^m\sum_{kl\in h}\mathrm{tr}(Z_j^TZ_{kl}Z_{kl}^TZ_j) &= \sum_{j=1}^m\sum_{kl\in\Delta(2)}\sum_{h\in H(w;kl)}\mathrm{tr}(Z_j^TZ_{kl}Z_{kl}^TZ_j)\\ = \binom{W-1}{w-1}\sum_{j=1}^m\sum_{kl\in\Delta(2)}\mathrm{tr}(Z_j^TZ_{kl}Z_{kl}^TZ_j),\end{aligned} \tag{A.6}$$

via a change in the order of summation. Since the treatment combinations in the design form an OA of strength two, by (A.1) and the definitions of Z_j and Z_{kl}, it can be seen that

$$Z_j^T Z_{kl} = 0, \quad \text{if } j = k \text{ or } j = l. \tag{A.7}$$

On the other hand, if j, k, l are distinct then recalling the definitions of Z_j, Z_{kl} and the $n^{jkl}_{\alpha\beta\gamma}$, $Z_j^T Z_{kl} = \sum\sum\sum n^{jkl}_{\alpha\beta\gamma} p_j(\alpha)\{p_k(\beta) \otimes p_l(\gamma)\}^T$, where the triple sum is over $0 \le \alpha \le s_j - 1, 0 \le \beta \le s_k - 1, 0 \le \gamma \le s_l - 1$. Hence, in this case, after some algebra using a property of the Kronecker product, one gets

$$\begin{aligned}\operatorname{tr}(Z_j^T Z_{kl} Z_{kl}^T Z_j) = \sum\sum\sum\sum\sum\sum n^{jkl}_{\alpha\beta\gamma} n^{jkl}_{\alpha^*\beta^*\gamma^*} [p_j(\alpha)^T p_j(\alpha^*)] \\ \times [p_k(\beta)^T p_k(\beta^*)][p_l(\gamma)^T p_l(\gamma^*)],\end{aligned} \tag{A.8}$$

the six fold sum being over $0 \le \alpha, \alpha^* \le s_j - 1, 0 \le \beta, \beta^* \le s_k - 1, 0 \le \gamma, \gamma^* \le s_l - 1$. By (A.1), $P_j^T P_j = s_j I_j - 1_j 1_j^T$, where I_j is the identity matrix of order s_j as in Section 2. Hence $[p_j(\alpha)^T p_j(\alpha^*)] = s_j \delta(\alpha, \alpha^*) - 1$, where $\delta(\alpha, \alpha^*)$ is Kronecker delta. Using similar expressions for the other terms in the right hand side of (A.8), it follows that

$$\operatorname{tr}(Z_j^T Z_{kl} Z_{kl}^T Z_j) = \text{constant} + \phi(jkl), \tag{A.9}$$

when j, k, l are distinct. The algebra underlying the passage from (A.8) to (A.9) uses (2.2) and the fact that the design is represented by an OA of strength two. By (A.7) and (A.9),

$$\sum_{j=1}^{m} \sum_{kl \in \Delta(2)} \operatorname{tr}(Z_j^T Z_{kl} Z_{kl}^T Z_j) = \text{constant} + \sum\sum\sum \phi(jkl), \tag{A.10}$$

where the triple sum is over j, k, l such that $1 \le j \le m$, $kl \in \Delta(2)$ and j, k, l are distinct. Recalling the definition of $\Delta(3)$, it is not hard to see that this triple sum equals $3 \sum_{jkl \in \Delta(3)} \phi(jkl)$. Hence from (A.6) and (A.10), one gets

$$\sum_{h \in H(w)} \sum_{j=1}^{m} \sum_{kl \in h} \operatorname{tr}(Z_j^T Z_{kl} Z_{kl}^T Z_j) = \text{constant} + 3 \binom{W-1}{w-1} \sum_{jkl \in \Delta(3)} \phi(jkl). \tag{A.11}$$

We next consider the sum of the last term in (A.5) over $h \in H(w)$. Analogously to (A.6),

$$\begin{aligned}&\sum_{h \in H(w)} \sum_{jk \in h} \sum_{lu \in h} \operatorname{tr}(Z_{jk}^T Z_{lu} Z_{lu}^T Z_{jk}) \\ &= \binom{W-1}{w-1} \sum_{jk \in \Delta(2)} \operatorname{tr}(Z_{jk}^T Z_{jk} Z_{jk}^T Z_{jk}) \\ &\quad + \binom{W-2}{w-2} \sum_{jk \in \Delta(2)} \sum_{lu (\ne jk) \in \Delta(2)} \operatorname{tr}(Z_{jk}^T Z_{lu} Z_{lu}^T Z_{jk}).\end{aligned} \tag{A.12}$$

Since the treatment combinations in the design form an OA of strength two, by (A.1) and the definition of Z_{jk},

$$Z_{jk}^T Z_{jk} = N I_{jk}^*, \tag{A.13}$$

for every $jk \in \Delta(2)$, where I_{jk}^* is the identity matrix of order $(s_j - 1)(s_k - 1)$. Turning to the second term in the right hand side of (A.12), observe that any two distinct members jk and lu of $\Delta(2)$ have either exactly one or no symbol in common. Hence, recalling the definitions of $\Delta(3)$ and $\Delta(4)$, after some algebra one gets

$$\begin{aligned}
&\sum_{jk\in\Delta(2)} \sum_{lu(\neq jk)\in\Delta(2)} \mathrm{tr}(Z_{jk}^T Z_{lu} Z_{lu}^T Z_{jk}) \\
&= 2 \sum_{jkl\in\Delta(3)} [\mathrm{tr}(Z_{jk}^T Z_{kl} Z_{kl}^T Z_{jk}) + \mathrm{tr}(Z_{jk}^T Z_{jl} Z_{jl}^T Z_{jk}) + \mathrm{tr}(Z_{jl}^T Z_{kl} Z_{kl}^T Z_{jl})] \\
&+ 2 \sum_{jklu\in\Delta(4)} [\mathrm{tr}(Z_{jk}^T Z_{lu} Z_{lu}^T Z_{jk}) + \mathrm{tr}(Z_{jl}^T Z_{ku} Z_{ku}^T Z_{jl}) + \mathrm{tr}(Z_{ju}^T Z_{kl} Z_{kl}^T Z_{ju})].
\end{aligned} \tag{A.14}$$

Steps similar to but more elaborate than those in the derivation of (A.9) yield

$$\mathrm{tr}(Z_{jk}^T Z_{kl} Z_{kl}^T Z_{jk}) = \text{constant} + (s_k - 2)\phi(jkl), \tag{A.15}$$

for $jkl \in \Delta(3)$, and

$$\begin{aligned}
\mathrm{tr}(Z_{jk}^T Z_{lu} Z_{lu}^T Z_{jk}) = {} & \text{constant} + \phi(jklu) \\
& -[\phi(jkl) + \phi(jku) + \phi(jlu) + \phi(klu)],
\end{aligned} \tag{A.16}$$

for $jklu \in \Delta(4)$. Since

$$\sum_{jklu\in\Delta(4)} [\phi(jkl) + \phi(jku) + \phi(jlu) + \phi(klu)] = (m-3) \sum_{jkl\in\Delta(3)} \phi(jkl), \tag{A.17}$$

by (A.15), (A.16), and similar expressions for other terms in the right hand side of (A.14),

$$\begin{aligned}
&\sum_{jk\in\Delta(2)} \sum_{lu(\neq jk)\in\Delta(2)} \mathrm{tr}(Z_{jk}^T Z_{lu} Z_{lu}^T Z_{jk}) \\
&= \text{constant} + 2 \sum_{jkl\in\Delta(3)} (s_j + s_k + s_l - 3m + 3)\, \phi(jkl) + 6 \sum_{jklu\in\Delta(4)} \phi(jklu).
\end{aligned} \tag{A.18}$$

From (A.5), (A.11)−(A.13), (A.18) and (2.1), the truth of (2.4), with E_w^* as in (2.3), follows.

Proof of Theorem 2.1. For any $jkl \in \Delta(3)$, let $n(jkl)$ be a vector, of order $s_j s_k s_l$, with elements $n^{jkl}_{\alpha\beta\gamma}(0 \le \alpha \le s_j - 1, 0 \le \beta \le s_k - 1, 0 \le \gamma \le s_l - 1)$, arranged lexicographically. Then by (2.6), (2.7) and the definition of $B(jkl)$, standard operations with Kronecker product show that

$$B(jkl) = \nu^{-1} n(jkl)^T \left[(s_j I_j - 1_j 1_j^T) \otimes (s_k I_k - 1_k 1_k^T) \otimes (s_l I_l - 1_l 1_l^T) \right] n(jkl).$$

Since the treatment combinations in the design form an OA of strength two, recalling (2.2) one gets

$$B(jkl) = \text{constant} + \nu^{-1} \phi(jkl). \tag{A.19}$$

Hence (2.11) yields

$$B_3 = \text{constant} + \nu^{-1} \sum_{jkl \in \Delta(3)} \phi(jkl). \tag{A.20}$$

In a similar manner,

$$\begin{aligned} B_4 &= \text{constant} + \nu^{-1} \sum_{jklu \in \Delta(4)} [\phi(jklu) - \{\phi(jkl) + \phi(jku) + \phi(jlu) + \phi(klu)\}] \\ &= \text{constant} + \nu^{-1} \Big(\sum_{jklu \in \Delta(4)} \phi(jklu) - (m-3) \sum_{jkl \in \Delta(3)} \phi(jkl) \Big), \end{aligned} \tag{A.21}$$

using (A.17). From (2.3) and (A.19)−(A.21), the truth of (2.10) follows.

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