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Statistics & Probability Letters 75 (2005) 203–210

STATISTICS & PROBABILITY LETTERS

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An approach for studying aliasing relations of mixed fractional factorials based on product arrays☆

Abhyuday Mandal

Department of Statistics, University of Georgia, Athens, GA 30602-1952, USA

Received 18 November 2004; received in revised form 6 May 2005; accepted 21 May 2005
Available online 12 July 2005

Abstract

Mixed-level fractional factorial designs are commonly used in industries but its aliasing relations have not been studied in full rigor. These designs take the form of a product array. Aliasing patterns of mixed level factorial designs, in the form of product arrays, are discussed here.

Keywords: Factorial design; Fractional factorial design; Product arrays

1. Introduction

Two- and three-level factorial and fractional factorial designs are widely used in industrial experimentations and are discussed in detail in design of experiments textbooks (Box et al., 1978; Cochran and Cox, 1950). The literature on symmetric designs is already voluminous. For example, the theory of regular fractions for symmetric factorials is given by Dey and Mukerjee (1999). Wu and Hamada (2000) devote a full chapter in their applied design of experiment textbook, on analysis techniques for mixed-level factorial plans. Although these are important designs, their aliasing patterns have not been studied explicitly.

Mixed-levels designs typically occur when there are both qualitative and quantitative factors in the experiment, and the qualitative factors have more than two levels and the quantitative factors

☆The research is supported by NSF grant DMS-0305996.
E-mail address: amandal@stat.uga.edu.

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doi:10.1016/j.spl.2005.05.021

have two levels. Consider an experiment by Hale-Bennett and Lin (1997) and reported in Wu and Hamada (2000) that was performed to improve a painting process of charcoal grill parts. A mixed-level 36-run design was used to study six factors: three of them (A, B, C) were at two levels and the other three (D, E, F) were at three levels. It is a $2^{3-1} \times 3^{3-1}$ design which consists of $4 \times 9 = 36$ runs and is a "product" of a 4-run 2^{3-1} and a 9-run 3^{3-1} design. Now it is not evident that the factorial effect ABD^2E is same as that of $ABDE^2$. If all the factors of a factorial effect are at two-levels, AB for example, a modulo 2 operation should be performed. Similarly, modulo 3 operations are used when all of them are have three level, as in the case for DE^2. But what about $ABDE^2$? It is not obvious whether modulo 2 or modulo 3 operations should be done in calculating the aliasing relationship of a mixed level factorial effect. In fact, there is no simple answer to this question, as will be clear from the discussions of Section 2.

In Section 2, we develop the general theory for $s_1^{n_1} \times s_2^{n_2}$ factorial designs and illustrate it in the context of a $2^3 \times 3^3$ design. In Section 3, we discuss $s_1^{n_1-k_1} \times s_2^{n_2-k_2}$ factorial designs and discuss the Paint experiment as an example of $2^{3-1} \times 3^{3-1}$ design. The results obtained here pertain to product arrays only.

2. $s_1^{n_1} \times s_2^{n_2}$ factorial designs

An experiment involving n_1 factors each at s_1 levels and n_2 factors each at s_2 levels is an $s_1^{n_1} \times s_2^{n_2}$ asymmetrical factorial experiment. Suppose the levels of the s_i-level factor are coded as s_i elements of Galois field $GF(s_i)$ where s_i is a prime or prime power. With levels as $0, 1, \ldots, s_i - 1$, a typical treatment combination, i.e., a combination of the levels of the $n_1 + n_2 = n$ factors will be represented by an ordered n-tuple $i_1 \ldots i_{n_1} j_1 \ldots j_{n_2}$ where $i_k \in \{0, 1, \ldots, s_1 - 1\}, 1 \leqslant k \leqslant n_1$ and $j_k \in \{0, 1, \ldots, s_2 - 1\}, 1 \leqslant k \leqslant n_2$. Clearly, altogether there are $s_1^{n_1} s_2^{n_2}$ treatment combinations.

In what follows, (a, b) and $(a', b')'$ will be used interchangeably for the sake of notational simplicity where a and b are column vectors of dimension n_1 and n_2, respectively.

A treatment contrast L is said to belong to the pencil (a, b) if it is of the form

$$L = \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} l(i,j) \left\{ \sum_{(x,y) \in V_{i,j}(a,b)} \tau(x, y) \right\}, \tag{1}$$

where $V_{i,j}(a, b) = \{(x, y) = (x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2},)' : a'x = \alpha_i, b'y = \beta_j\}$, $0 \leqslant i \leqslant s_1 - 1$, $0 \leqslant j \leqslant s_2 - 1$; the effect of a treatment combination represented by (x, y) will be denoted by $\tau(x, y)$ and $l(i,j)$'s are real numbers, not all zeros, satisfying

$$\sum_{i=0}^{s_1-1} l(i,j) = \sum_{j=0}^{s_2-1} l(i,j) = 0. \tag{2}$$

In other words, a treatment contrast L belongs to (a, b) if for all (x, y) belonging to the same $V_{i,j}(a, b)$, the coefficient of $\tau(x, y)$ in L is also the same.

In general, consider any two pencils (a, b) and (a^*, b^*). These two pencils are distinct if a is distinct from a^* and b is distinct from b^*, in the sense of symmetric factorial designs. Recall that, in symmetric fractions, pencils with proportional entries are considered as identical.

For a $s_1^{n_1} \times s_2^{n_2}$ product array, for the $s_i^{n_i}$ factorial part, there are $(s_i^{n_i} - 1)/(s_i - 1)$ distinct pencils which involve only s_i-level factors. The distinct pencils involving both s_1- and s_2-level factors are given by the products of those two sets of pencils involving only s_1- or only s_2-level factors. A simple counting of degrees of freedom justifies this formulation. Recall that the total number of factorial effects involving only s_i-level factors is $(s_i^{n_i} - 1)/(s_i - 1)$, each with $(s_i - 1)$ d.f.. The interactions involving both s_1- and s_2-level factors are given by products of the other two sets of pencils, i.e., there are $(s_1^{n_1} - 1)/(s_1 - 1) \times (s_2^{n_2} - 1)/(s_2 - 2)$ pencils of this kind, each with $(s_1 - 1)(s_2 - 1)$ d.f.. Thus, the above description accounts for

$$\frac{s_1^{n_1} - 1}{s_1 - 1}(s_1 - 1) + \frac{s_2^{n_2} - 1}{s_2 - 1}(s_2 - 1) + \frac{s_1^{n_1} - 1}{s_1 - 1}\frac{s_2^{n_2} - 1}{s_2 - 1}(s_1 - 1)(s_2 - 1) = s_1^{n_1}s_2^{n_2} - 1$$

d.f. which agrees with the fact that there are $s_1^{n_1}s_2^{n_2}$ in all.

Following Bose (1947), Dey and Mukerjee (1999) gave the definition for treatment contrasts belonging to factorial effects for the general case of an $s_1 \times \cdots \times s_n$ factorials. The next two results link pencils with factorial effects.

Result 2.1. (a) *Treatment contrasts belonging to distinct pencils are orthogonal to each other.* (b) *Let (a, b) be a pencil such that $a_i \neq 0$ if $i \in \{i_1, \ldots, i_g\}$, and $= 0$ otherwise, $b_j \neq 0$ if $j \in \{i_1, \ldots, i_h\}$, and $= 0$ otherwise*, where $1 \leqslant i_1 < \cdots < i_g \leqslant n_1$, $1 \leqslant j_1 < \cdots < i_h \leqslant n_2$ *and* $1 \leqslant g \leqslant n_1$, $1 \leqslant h \leqslant n_2$. *Then any treatment contrast belonging to (a, b) also belongs to the factorial effect $F_{i_1} \ldots F_{i_g} F'_{j_1} \ldots F'_{j_h}$.*

Example. Let us consider the $2^3 \times 3^3$ full factorial design with two-level factors A, B, C and three-level factors D, E, F. The levels of A, B, C are denoted by 0 and 1, and those of D, E, F are denoted by 0, 1 and 2. Then a typical treatment combination, i.e., the combination of the levels of six factors will be represented by $x = (a, b, c, d, e, f)$, where $a, b, c \in \{0, 1\}$ and $d, e, f \in \{0, 1, 2\}$. For example, the factorial effect $ABDE^2$ is denoted by $(a, b, c, d, e, f) \equiv (1, 1, 0, 1, 2, 0)$. Clearly there are $2^3 \times 3^3 = 216$ possible treatment combinations.

The pencils involving only the two-level factors or only the three-level factors can be described as usual. Thus the pencil AB is given by the contrasts between the two sets of treatment combinations for which $a + b = 0$ or $1 \bmod 2$. More explicitly, these two sets are $\{x : x = (a, b, c, d, e, f), a + b = 0 \bmod 2\}$ and $\{x : x = (a, b, c, d, e, f), a + b = 1 \bmod 2\}$. Clearly, there are 108 treatment combinations in each of these sets, e.g., the first set consists of the treatment combinations $(0, 0, c, d, e, f)$ and $(1, 1, c, d, e, f)$, where $c \in \{0, 1\}$ and $d, e, f \in \{0, 1, 2\}$, leading to $54 + 54 = 108$ treatment combinations in all. In a similar manner, the pencil DEF^2, involving exclusively the three-level factors, is given by contrasts among three sets of treatment combinations for which $d + e + 2f = 0, 1$ or 2 mod 3. As before, there are $8 \times 9 = 72$ treatment combinations in each of these sets. It is clear that any pencil involving A, B, C will have 1 d.f. while any pencil involving only D, E, F will have 2 d.f.

Now consider the interactions that involve both two- and three-level factors. Recall that there are seven pencils A, B, C, AB, AC, BC and ABC involving only the two-level factors. Similarly there are 13 distinct pencils D, E, F, DE, $DE^2, \ldots, DE^2F^2$ involving only the three-level factors. The pencils representing interactions that involve both two- and three-level factors are given by the products of these two sets of pencils, i.e., there are $7 \times 13 = 91$ pencils of this kind, namely, AD, AE, $\ldots$, ADE^2F^2, BD, $BE, \ldots, BDE^2F^2, \ldots, ABCD$, $ABCE, \ldots, ABCDE^2F^2$. Each of these

91 pencils carries 2 d.f.. Clearly, for example, AD and AD^2 mean the same thing in this formulation (so we write only AD). Similarly $ABDE^2 = ABD^2E = AB(DE^2)^2$. Taking care of the seven pencils involving only the two-level factors and the 13 pencils involving only the three-level factors, the above description accounts for $7 \times 1 + 13 \times 2 + 91 \times 2 = 215$ d.f., which agrees with the fact that there are $2^3 \times 3^3 = 216$ treatment combinations in all.

How does one actually define contrasts belonging to pencils as considered in the last paragraph? Consider, for example, the pencil $ABDE^2$. For $i = 0, 1$ and $j = 0, 1, 2$, define $V_{i,j} = V_{i,j}(110, 120) = \{x : x = (a, b, c, d, e, f), a + b = i \bmod 2, d + 2e = j \bmod 3\}$. Note that $a + b = i \bmod 2$ corresponds to AB, and $d + 2e = j \bmod 3$ corresponds to DE^2. Clearly, each of the six sets $V_{i,j}$ has cardinality $4 \times 9 = 36$. Let $T(i,j)$ be the total of the treatment effects for the treatment combinations in $V_{i,j}$. Then a typical contrast belonging to the pencil $ABDE^2$ will be of the form $\sum_i \sum_j l(i,j)T(i,j)$, where the scalars $l(i,j)$, not all zeros, must satisfy $l(0,j) + l(1,j) = 0$ for every j and $l(i,0) + l(i,1) + l(i,2) = 0$ for every i. Thus there will be two such independent treatment contrasts, namely, $L_1 = T(0,0) - T(1,0) - T(0,2) + T(1,2)$ and $L_2 = T(0,0) - T(1,0) - 2T(0,1) + 2T(1,1) + T(0,2) - T(1,2)$.

3. $s_1^{n_1-k_1} \times s_2^{n_2-k_2}$ fractional factorial designs

A regular fraction of an s^n symmetrical factorial, where s ($\geqslant 2$) is a prime or prime power, is specified by any k ($1 \leqslant k < n$) linearly independent pencils, say $b^{(1)}, \ldots, b^{(k)}$, and consists of treatment combinations z satisfying $Bz = c$, where B is a $k \times n$ matrix with rows $(b^{(i)})'$, $1 \leqslant i \leqslant k$, and c is a fixed $k \times 1$ vector over $GF(s)$. The specific choice of c is inconsequential. Hence, without loss of generality, it is assumed that $c = 0$, the $k \times 1$ null vector over $GF(s)$. Then a regular fractional factorial plan is given by, say, $d(B) = \{z : Bz = 0\}$.

In the same line, for a $s_1^{n_1} \times s_2^{n_2}$ design, a regular fractional factorial plan, $s_1^{n_1-k_1} \times s_2^{n_2-k_2}$ is given by $d(B) = \{z : Bz = 0\} = \{(x, y) : B_1 x = 0, B_2 y = 0\}$ where

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}.$$

Note that $d(B_i)$ gives a regular $s_i^{n_i-k_i}$ fractional factorial plan. For a symmetric fractional factorial, a pencil is called a defining pencil if it belongs to the row space of B. Equivalently, a defining pencil of a $s_1^{n_1-k_1} \times s_2^{n_2-k_2}$ design is of the form (b_1, b_2) where b_i is a defining pencil of $s_i^{n_i-k_i}$.

Consider now any defining pencil (a, b). Then $a' = \lambda' B_1$ and $b' = \xi' B_2$ for suitable λ and ξ with entries from $GF(s_1)$ and $GF(s_2)$, respectively. Now it is not difficult to see that $d(B) \subset V_{0,0}(a, b)$. Recalling the definition of a treatment contrast, the following result is evident.

Result 3.1. *No treatment contrast belonging to any defining pencil is estimable in $d(B)$.*

Two pencils are aliases of each other if their difference belongs to the row space of B. Let $\mathscr{C}$ be the set of distinct pencils which are not defining pencils. Then we get the following Lemma.

Lemma 3.1. *Let the pencils $(a, b), (a^*, b^*) \in \mathscr{C}$ be aliases of each other and L and L^* be the treatment contrasts belonging to (a, b) and (a^*, b^*), respectively. Then the parts of L and L^*, which involve only the treatment combinations included in $d(B)$, are identical.*

Let $L(B)$ be the part of L that involves only the treatment combination involved in the fraction $d(B)$ and is often called the relevant part of L. Then the relevant parts of corresponding contrasts belonging to pencils that are aliases of each other, are identical. Let $V_{i,j}((a,b),B) = V_{i,j}(a,b) \cap d(B)$. Then for any pencil $(a,b) \in \mathscr{C}$ and for its alias set $\mathscr{A}$, we get the following theorem.

Theorem 3.1. *For $(a,b) \in \mathscr{A}$, consider the corresponding treatment contrast given in Eq.* (1). *Then*

$$\sum_{(a,b)}\left[\sum_{i=0}^{s_1-1}\sum_{j=0}^{s_2-1} l(i,j)\left\{\sum_{(x,y)\in V_{i,j}(a,b)} \tau(x,y)\right\}\right] = s_1^{k_1}s_2^{k_2}\sum_{i=0}^{s_1-1}\sum_{j=0}^{s_2-1} l(i,j)\left\{\sum_{(x,y)\in V_{i,j}((a,b),B)} \tau(x,y)\right\}, \tag{3}$$

where $\sum_{(a,b)}$ denote the sum over all $(a,b) \in \mathscr{A}$.

To prove this theorem, we need the following Lemma.

Lemma 3.2. *Consider any pencil $(a,b) \in \mathscr{C}$ and let $\mathscr{A}$ denote its alias set. Let*

$$\phi_{i,j}((a,b)(x,y)) = \begin{cases} 1 & \text{if } a'x = \alpha_i, b'y = \beta_j, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Then for every treatment combination (x,y) and every (i,j), $0 \leqslant i \leqslant s_1 - 1, 0 \leqslant j \leqslant s_2 - 1$,

$$\sum_{(a,b)} \phi_{i,j}((a,b),(x,y)) = \begin{cases} s_1^{k_1}s_2^{k_2} & \text{if } (x,y) \in V_{i,j}((a,b),B), \\ 0 & \text{if } (x,y) \in d(B) - V_{i,j}((a,b),B), \\ s_1^{k_1-1}s_2^{k_2-1} & \text{if } (x,y) \notin d(B). \end{cases} \tag{5}$$

Proof. A pencil in $\mathscr{A}$ is of the form (p,q) where $p = a + B_1'\lambda$ and $q = b + B_2'\xi$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k_1})'$, $\lambda_i \in GF(s_i)$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_{k_2})'$, $\xi_j \in GF(s_2)$. For fixed (x,y) and (i,j), we get $\sum_{(a,b)}\phi_{i,j}((a,b),(x,y)) = \#\{\lambda = (\lambda_1, \ldots, \lambda_{k_1})' : x + \lambda' B_1 x = \alpha_i; \xi = (\xi_1, \ldots, \xi_{k_2})' : b'y + \xi' B_2 y = \beta_j, \lambda_i \in GF(s_i), \xi_j \in GF(s_2)\ \forall i,j\}$ where # denotes the cardinality of a set.

(i) If $(x,y) \in V_{i,j}(a,b)$ then $a'x + \lambda' B_1 x = \alpha_i$ for all $k_1 \times 1$ vectors over $GF(s_1)$ and $b'y + \xi' B_2 y = \beta_j$ for all $k_2 \times 1$ vectors over $GF(s_2)$. Hence the RHS of (5) is $s_1^{k_1}s_2^{k_2}$.

(ii) If $(x,y) \in d(B) - V_{i,j}((a,b),B)$, then $B_1 x = 0$, $B_2 = 0$. Also, $a'x \neq \alpha_i$ and/or $b'y \neq \beta_j$. Then $\sum_{(a,b)}\phi_{i,j}((a,b),(x,y)) = \#\{(\lambda,\xi) : a'x = \alpha_i, b'y = \beta_j\} = 0$.

(iii) If $(x,y) \notin d(B)$, then $B_1 x \neq 0$, $B_2 y \neq 0$. Trivially $a'x + \lambda' B_1 x = \alpha_i$ iff $(B_1 x)'\lambda = \alpha_i - a'x$. Since $B_1 x \neq 0$, exactly as in the proof of Lemma 2.1, one can freely choose $(\lambda_2, \ldots, \lambda_{k_1-1})$ in $s_1^{k_1-1}$ ways to satisfy the above equation. Similarly $b'y + \xi' B_2 y = \beta_j$ gives $s_2^{k_2-1}$ choices of ξ_l's. Combining the values of λ_k's and ξ_l's, the result follows.

Proof of Theorem 3.1. Let Ω denote the set of all $s_1^{n_1}s_2^{n_2}$ treatment combinations. Using Lemma 3.2 and the indicator variable $\phi_{i,j}((a,b)(x,y))$ in (4),

$$\sum_{(a,b)}\left[\sum_{i=0}^{s_1-1}\sum_{j=0}^{s_2-1} l(i,j)\left\{\sum_{(x,y)\in V_{i,j}(a,b)} \tau(x,y)\right\}\right]$$

$$= \sum_{(a,b)} \left[\sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} l(i,j) \left\{ \sum_{(x,y)\in\Omega} \phi_{i,j}((a,b)(x,y))\tau(x,y) \right\} \right]$$

$$= \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} l(i,j) \left\{ \sum_{(x,y)\in\Omega} \left[\sum_{(a,b)} \phi_{i,j}((a,b)(x,y)) \right] \tau(x,y) \right\}$$

$$= s_1^{k_1} s_2^{k_2} \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} l(i,j) \left\{ \sum_{(x,y)\in V_{i,j}((a,b),B)} \tau(x,y) \right\},$$

since $\sum l(i,j) = 0$.

The RHS of (3) is a contrast involving only the treatment combinations included in $d(B)$. Therefore the RHS and hence the LHS of (3) will be estimable using the plan $d(B)$. In other words, while pencils belonging to the same alias set, are confounded with one another (Lemma A.3), the sum of corresponding contrasts, belonging to such pencils is estimable in $d(B)$. Thus any treatment contrast belonging to a pencil (a, b) which is not a defining pencil is estimable in $d(B)$ if and only if corresponding contrasts belonging to all other pencils that are aliased with (a, b) are ignorable.

We say that a pencil is estimable in $d(B)$ if so is every treatment contrast belonging to it. Similarly, if every treatment contrast belonging to a pencil is ignorable, then the pencil itself is called ignorable. Hence the following result is immediate.

Result 3.2. *A pencil b, which is not a defining pencil, is estimable in $d(B)$ if and only if all other pencils that are aliased with b are ignorable.*

Example. Now consider the fractional factorial design used for the Paint experiment. This kind of fraction treats the two- and three-level factors separately, leading to a product array.

The defining relation of the $2^{3-1} \times 3^{3-1}$ design can be obtained from those of its two component designs: $I = ABC$ and $I = DEF^2$. So we decide to include the treatment combinations $x = (a, b, c, d, e, f)$ satisfying $a + b + c = 0 \bmod 2$ and $d + e + 2f = 0 \bmod 3$. There are four such choices of (a, b, c) and nine such choices of (d, e, f). Combining these, we will have $4 \times 9 = 36$ treatment combinations in our plan which will be in the from of product array. The alias sets will again be of three types:

Type I (involving only two-level factors arising from $I = ABC$): These are $A = BC$, $B = AC$, $C = AB$, each carrying 1 d.f.

Type II (involving only three-level factors arising from $I = DEF^2$): there will be four such alias sets, each carrying 2 d.f. these are $D = DE^2F = EF^2$; $E = DF^2 = DE^2F^2$; $F = DE = DEF$ and $DE^2 = DF = EF$.

Type III (involving the "mixed" pencils discussed earlier): These are obtained by combining each type I alias set with each type II alias set, e.g., a typical alias set of type III will be $AD = ADE^2F = AEF^2 = BCD = BCDE^2 = BCEF^2$. There will be $3 \times 4 = 12$ such alias sets each carrying 2 d.f.

For any pencil in a type III alias set, it is not hard to see that each set $V_{i,j}$ corresponding to that pencil will contain six of the treatment combinations included in our fraction. To see this, consider the pencil $BCDE^2F$. A treatment combination $x=(a,b,c,d,e,f)$ in our fraction will then belong to the corresponding $V_{i,j}$ if it satisfies $b+c=i \bmod 2$ and $d+2e+f=j \bmod 3$, in addition to satisfying $a+b+c=0 \bmod 2$ and $d+e+2f=0 \bmod 3$ needed for inclusion in the fraction. Now the first and third of the equations just mentioned yield two solutions for (a,b,c) while the second and fourth of these equations yield three solutions for (d,e,f). Combining these, we get six solutions altogether.

4. Summary

The designs discussed here are called product arrays. These are quite common in robust parameter designs and are named *cross arrays* (*inner-outer array* in Taguchi's terminology). Here only $s_1^{n_1} \times s_2^{n_2}$ factorials are discussed, although with heavier notations, and without any significant conceptual change, it is possible to obtain general results for $s_1^{n_1} \times s_2^{n_2} \times \cdots s_m^{n_m}$ factorials.

Acknowledgements

The author is thankful to Professor Rahul Mukerjee and Professor C. F. Jeff Wu for their comments and suggestions.

Appendix A

The following two lemmas are immediate.

Lemma A.1. *Let $(a,b)=(a_1,\ldots,a_{n_1},b_1,\ldots,b_{n_2})'$ be any fixed nonnull vector where $a_i \in GF(s_1)$ and $b_j \in GF(s_2)$. Then each of the sets*

$$V_{i,j}(a,b)=\{(x,y)=(x_1,\ldots,x_{n_1},y_1,\ldots,y_{n_2},)' : a'x=\alpha_i, b'y=\beta_j\} \tag{A.1}$$

$0 \leqslant i \leqslant s_1-1$, $0 \leqslant j \leqslant s_2-1$ *has cardinality* $s_1^{n_1-1}s_2^{n_2-1}$.

Lemma A.2. *If $(a^{(1)},b^{(1)})$ and $(a^{(2)},b^{(2)})$ are distinct pencils, then for every $(i,j),(i',j')$, $(0 \leqslant i,i' \leqslant s_1-1, 0 \leqslant j,j' \leqslant s_2-1)$, the set $V_{i,j}(a^{(1)},b^{(1)}) \cap V_{i',j'}(a^{(2)},b^{(2)})$ has cardinality $s_1^{n_1-2}s_2^{n_2-2}$.*

Proof of Result 2.1. (a) Let L and L^* be the treatment contrasts belonging to (a,b) and (a^*,b^*), respectively (Refer Eq. (1)). Consider the scalar product S of the coefficient vectors of L and L^*. Observe that, for any (i,j) and (k,l), if $(x,y) \in V_{i,j}(a,b) \cap V_{i,j}(a^*,b^*)$, then the contribution of $\tau(x,y)$ to S equals $l(i,j)l^*(k,l)$. Hence S equals $\sum\sum l(i,j)l^*(k,l)\#\{V_{i,j}(a,b) \cap V_{k,l}(a^*,b^*)\}$ which is zero by Lemma A.2.

Proof of Result 2.1. (b) Without loss of generality, let $i_1=1,\ldots,i_g=g$ and $j_1=1,\ldots,j_h=h$. Then $a_1,\ldots,a_g$ are nonzero while $a_{g+1}=\cdots=a_{n_1}=0$ and $b_1,\ldots,b_h$ are nonzero while $b_{h+1}=\cdots=b_{n_2}=0$, so that $V_{i,j}(a,b)=\{(x,y): \sum_{k=1}^{g} a_k x_k=\alpha_i, \sum_{l=1}^{h} b_l y_l=\beta_j\}$, $0 \leqslant i \leqslant s_1-1$ and $0 \leqslant j \leqslant s_2-1$. Recalling the definition of a treatment contrast L in Eqs. (1) and (2),, it is easy to see that for and (x,y), the coefficient of $\tau(x,y)$ in L depends on (x,y) only through

$x_1, \ldots, x_g$ and $y_1, \ldots, y_h$. Writing $\bar{l}(x_1, \ldots, x_g, y_1, \ldots, y_h)$ for the coefficient of $\tau(x, y)$ in L, one gets $\bar{l}(x_1, \ldots, x_g, y_1, \ldots, y_h) = l(i,j)$, if for $0 \leqslant i \leqslant s_1 - 1$ and $0 \leqslant j \leqslant s_2 - 1$,

$$\sum_{k=1}^{g} a_k x_k = \alpha_i, \quad \sum_{l=1}^{h} b_l y_l = \beta_j. \tag{A.2}$$

Now, as $a_1 \neq 0$, the quantity $\sum_{k=1}^{g} a_k x_k$ equals each of $\alpha_0, \alpha_1, \ldots, \alpha_{s_1 - 1}$ once as x_1 assumes all possible values over $GF(s_1)$, each exactly once, for any fixed $x_2, \ldots, x_g, y_1, \ldots, y_h$. Hence by (A.2), we get $\sum_{x_1 \in GF(s_1)} \bar{l}(x_1, \ldots, x_g, y_1, \ldots, y_h) = 0$, for any fixed $x_2, \ldots, x_g, y_1, \ldots, y_h$. Similar arguments for other x_k and y_l's complete the proof. $\square$

Proof of Lemma 3.1. It is enough to show that $V_{i,j}((a,b), B) = V_{i,j}((a^*, b^*), B)\ \forall i,j$. Since (a,b) and (a^*, b^*) are aliases of each other, we have $a - a^* = B_1' \lambda$ and $b - b^* = B_2' \xi$ for suitable $\lambda \in GF(s_1)$ and $\xi \in GF(s_2)$. Now,

$$\begin{aligned}
V_{i,j}((a,b), B) &= \{(x,y) : a'x = \alpha_i, b'y = \beta_j, B_1 x = 0, B_2 y = 0\} \\
&= \{(x,y) : (a^* + B_1'\lambda)'x = \alpha_i, (b^* + B_2'\xi)'y = \beta_j, B_1 x = 0, B_2 y = 0\} \\
&= \{(x,y) : a^{*\prime}x + \lambda' B_1 x = \alpha_i, b^{*\prime}y + \xi' B_2 y = \beta_j, B_1 x = 0, B_2 y = 0\} \\
&= \{(x,y) : a^{*\prime}x = \alpha_i, b^{*\prime}y = \beta_j, B_1 x = 0, B_2 y = 0\} \\
&= V_{i,j}((a^*, b^*), B).
\end{aligned}$$

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